$\vec{u} = \langle 3, 2 \rangle$ ← some authors $\vec{u} = (3, 2)$ OR $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

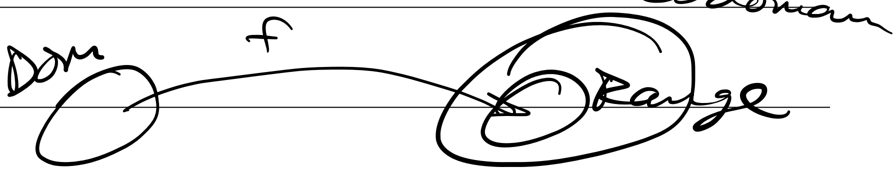
1.8 – Introduction to Linear Transformations

Definitions: $\mathbf{R}^2$ is the set of all ordered pairs, $\mathbf{R}^3$ is the set of all ordered triples, and $\mathbf{R}^n$ is the set of all ordered n-tuples. Elements of R^n are called **vectors**.

The **standard basis vectors** for R^n are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, …, $\mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. These can also be written as row vectors if convenient.

All vectors in $\mathbb{R}^n$ can be expressed as linear combinations of these:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$$



Definitions: A **function** f is a rule that associates with an input (an element of the **domain** of f) a unique output (an element of the **codomain** of f). The output is the **image** of the input, and the set of all images is called the **range** of f. Note that the range of f is a subset of the codomain of f.

$f(a) = b$ ← image of a under f

$a \in$ domain of f $b \in$ codomain of f

Consider $f(x) = |x|$ Domain: $\mathbb{R}$

type of critter → Codomain: $\mathbb{R}$

→ Range: $[0, \infty)$

set of critters

Transformations

Consider the system
$$4 = 2x - 3y + 5z$$
$$6 = 7x + 2y - z$$

In matrix form this is
$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The 2×3 matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix}$ maps vectors in R^3 (domain) into vectors in R^2 (codomain).

A is called a transformation matrix from R^3 to R^2. $T_A : R^3 \to R^2$

Definition: If T is a function with domain R^n and codomain R^m, then we say that T is a **transformation** from R^n to R^m or that T **maps** from R^n to R^m, which we denote by writing $T : R^n \to R^m$. In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n.

#5 Find the domain and codomain of the transformation defined by the matrix product.

a. $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

a) Dom: R^3
Codomain: R^2

b) Dom: R^2
Codomain: R^3

Definitions: A **matrix transformation w** $= A\mathbf{x}$ maps a vector $\mathbf{x} \in R^n$ to a vector $\mathbf{w} \in R^m$ by multiplying $\mathbf{x}$ on the left by A [which is an $m \times n$ matrix]. If $m = n$, then we call the transformation a **matrix operator**. A matrix transformation is denoted $T_A : R^n \to R^m$ or $\mathbf{w} = T_A(\mathbf{x})$ if we do not need to specify the domain and codomain. This can also be written in the form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

verbalized as "T_A maps $\mathbf{x}$ into $\mathbf{w}$." The matrix A is the **standard matrix** for the transformation.

#**8** Find the domain and codomain of the transformation T defined by the formula.

a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$

b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

a) Dom: R^4, Codomain: R^2

b) Dom: R^3, codomain: R^3 (operator)

#**12** Find the standard matrix for the transformation defined by the equations.

a. $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 - 2x_2 \\ 5x_1 - 7x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

b.
$$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

a) $A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$T(\vec{x})$ $T(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $A\vec{x}$

#13 Find the standard matrix for the transformation T defined by the formula.

a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

b. $T(x_1, x_2, x_3) = (4x_1 + x_2, x_1 + x_2)$ $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$

b) $\boxed{[T] = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$ a) $\boxed{[T] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}$

Definitions: The **zero transformation** from R^n to R^m, $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$, maps every vector in R^n to the zero vector in R^m. The **identity operator** $T_{I_n}(\mathbf{x}) = I_n(\mathbf{x}) = \mathbf{x}$ maps every vector in R^n to itself.

in down in column

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Theorem 1.8.1 Properties of Matrix Transformations

For every matrix A the matrix transformation $T_A : R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

a) $T_A(\mathbf{0}) = \mathbf{0}$

b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

These follow from properties of matrix arithmetic

Theorem 1.8.2 $T : R^n \to R^m$ is a matrix transformation if and only if ($\Longleftrightarrow$) the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

i) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

pf: ($\Rightarrow$) by Thm 1.8.1 (b) & (c)

($\Leftarrow$) Assume (i) & (ii) hold. The proof will be complete if there is an $m \times n$

matrix A such that $T_A(\vec{x}) = A\vec{x}$ $\overset{\text{for all}}{\forall}$ $\vec{x} \overset{\text{in}}{\in} R^n$

By (i) & (ii), $T(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_n\vec{u}_n)$

$$= k_1 T(\vec{u}_1) + k_2 T(\vec{u}_2) + \cdots + k_n T(\vec{u}_n)$$

for all scalars k_i and vectors $\vec{u}_i \in R^n$.

Let $A = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)]$,

where $\vec{e}_i$ is a standard basis vector for R^n.

Let $\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$. Then

$$A\vec{k} = k_1 T(\vec{e}_1) + k_2 T(\vec{e}_2) + \cdots + k_n T(\vec{e}_n),$$

a linear combination of the column vectors of A in which coeffs. are entries of $\vec{k}$, as guaranteed by Thm 1.3.1.

Then $A\vec{k} = T(k_1\vec{e}_1 + k_2\vec{e}_2 + \cdots + k_n\vec{e}_n)$

by linearity of T.

<u>**Definitions:**</u> The properties (i) and (ii) in the preceding theorem are called **linearity conditions**. A transformation that possesses these properties is a **linear transformation**.

<u>**Theorem 1.8.3**</u> Every linear transformation from R^n to R^m is a matrix transformation, and conversely every matrix transformation from R^n to R^m is a linear transformation.

Theorem 1.8.4 If $T_A : R^n \to R^m$ and $T_B : R^n \to R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

#22 Use Theorem 1.8.2 to show that T is a matrix transformation.

a. $T(x, y, z) = (x + y, y + z, x)$

b. $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

We need to show that i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

and ii) $T(k\vec{u}) = kT(\vec{u})$

a) i) Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3) \in R^3$

$\vec{u} + \vec{v} = (u_1+v_1, u_2+v_2, u_3+v_3)$

$T(\vec{u}+\vec{v}) = (u_1+v_1+u_2+v_2,\ u_2+v_2+u_3+v_3,\ u_1+v_1)$

$= (\underline{u_1+u_2} + \underline{v_1+v_2},\ \underline{u_2+u_3} + \underline{v_2+v_3},\ \underline{u_1} + \underline{v_1})$

$= (u_1+u_2, u_2+u_3, u_1) + (v_1+v_2, v_2+v_3, v_1)$

$= T(\vec{u}) + T(\vec{v})$ ✓

ii) Let $\vec{u}$ be as above and k be a scaler.

$k\vec{u} = (ku_1, ku_2, ku_3) \Rightarrow T(k\vec{u}) = (ku_1+ku_2,\ ku_2+ku_3,\ ku_1)$

$= k(u_1+u_2, u_2+u_3, u_1)$

$= kT(\vec{u})$ ✓

We could have: $T(k\vec{u}+\vec{v}) = kT(\vec{u}) + T(\vec{v})$

23. Use Theorem 1.8.2 to show that T is not a matrix transformation.

a. $T(x, y) = (x^2, y)$

b. $T(x, y, z) = (x, y, xz)$

$T(x, y) = (x+1, y)$

$T(\vec{u}+\vec{v}) = (u_1+v_1+1,\ u_2+v_2)$

$\neq$

$T(\vec{u}) + T(\vec{v}) = (u_1+1, u_2) + (v_1+1, v_2)$

$= (u_1+v_1+2, u_2+v_2)$

a) Let $\vec{u} = (u_1, u_2)$ and $k \in R$.

$k\vec{u} = (ku_1, ku_2)$ $\qquad$ $T(\vec{u}) = (u_1^2, u_2)$

$$T(k\vec{u}) = ((ku_1)^2, ku_2) = (k^2u_1^2, ku_2)$$
$$= k(ku_1^2, u_2)$$
$$\neq kT(\vec{u})$$

$$T(\vec{u}+\vec{v}) = ((u_1+v_1)^2, u_2+v_2) = (u_1^2 + 2u_1v_1 + v_1^2, u_2+v_2)$$

$$\neq T(\vec{u}) + T(\vec{v})$$

#28 The images of the standard basis vectors for R^3 are given for a linear transformation $T : R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$T_A(\vec{x}) = A\vec{x} = [T_A]\vec{x}$

$\hookrightarrow [T] = [T(\vec{e}_1) \mid T(\vec{e}_2) \mid T(\vec{e}_3)]$

$$[T] = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

why?

$$[T]\vec{e}_2 = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}$$

$T(2,0,1) = (5,2,8)$

Recall that if $\vec{w} = a\vec{u} + b\vec{v}$, then $T(\vec{w}) = aT(\vec{u}) + bT(\vec{v})$

Example: Find the standard matrix A for the linear transformation $T : R^2 \to R^3$ for which

So if $\vec{e}_1 = c\,\vec{u} + d\,\vec{v}$, then $T(\vec{e}_1) = cT(\vec{u}) + dT(\vec{v})$

$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$$

$T(\vec{u})$ $\qquad$ $T(\vec{v})$

and use it to compute $T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right)$.

To express $\vec{e}_1$ as a linear combination of $\vec{u}$ & $\vec{v}$, we have $\vec{e}_1 = a\vec{u} + b\vec{v}$.

System: $a\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + b\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

becomes $\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \left[\begin{array}{cc|c} u_1 & v_1 & 1 \\ u_2 & v_2 & 0 \end{array}\right]$

For $\vec{e}_2$, we have $\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We can solve these simultaneously:

$$\left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 2 & -5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$\qquad\qquad\qquad\qquad \vec{e}_1 \quad \vec{e}_2$

If we only did $\left[\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & -5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right]$

$5\vec{u} + 2\vec{v} = \vec{e}_1$ $\qquad$ $3\vec{u} + 1\vec{v} = \vec{e}_2$

$T(\vec{e}_1) = 5T(\vec{u}) + 2T(\vec{v})$

$= 5\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$

$T(\vec{e}_2) = 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 5 \\ -7 \\ 1 \end{bmatrix}$

$$T(\vec{e}_1) = \begin{bmatrix} 20 \\ -9 \\ 2 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix} = \left[\, T(\vec{e}_1) \mid T(\vec{e}_2) \,\right]$$

$$T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -47 \\ 24 \\ -5 \end{bmatrix}$$

Matrix operators on R^2

Reflection operators

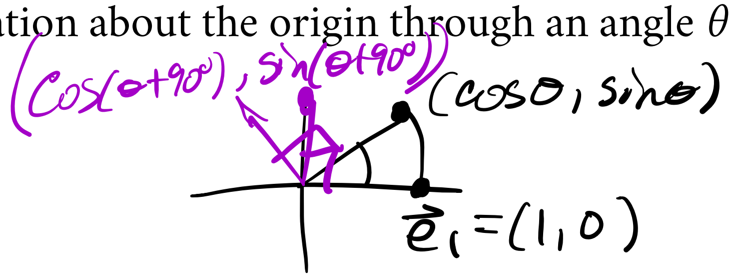
- Reflection about the x-axis: $T(x,\ y) = \ (x,\ -y)$, $T_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection about the y-axis: $T(x,\ y) = \ (-x,\ y)$, $T_A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection about the line $y = x$: $T(x,\ y) = (y,\ x)$, $T_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Projection operators

- Orthogonal projection onto the x-axis: $T(x,\ y) = \ (x,\ 0)$,
  $T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Orthogonal projection onto the y-axis: $T(x,\ y) = \ (0,\ y)$,
  $T_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Rotation operator

- Counterclockwise rotation about the origin through an angle θ:
  $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Matrix operators on R^3

Reflection operators

- Reflection about the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ -z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
- Reflection about the xz-plane: $T(x,\ y,\ z) = (x,\ -y,\ z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Reflection about the yz-plane: $T(x,\ y,\ z) = (-x,\ y,\ z)$,
$$T_A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Projection operators</u>

- Orthogonal projection onto the xy-plane: $T(x,\ y,\ z) = (x,\ y,\ 0)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- Orthogonal projection onto the xz-plane: $T(x,\ y,\ z) = (x,\ 0,\ z)$,
$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Orthogonal projection onto the yz-plane: $T(x,\ y,\ z) = (0,\ y,\ z)$,
$$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(age, height, weight, # children)
(in) (lbs)

Codomain

(2, 3)

(49, 73, 175, 2)

Range

(472, −15, 98, 2.7)

$T: \mathbb{R} \to \mathbb{R}^2$ such that $T(x) = (x, x)$

$\mathbb{R}^2$

Dom: $\mathbb{R}$

$\cancel{(2,3)}$

Codomain: $\mathbb{R}^2$

$(5,5)$

Range is of form (a, a)